LITERATURE CITED

1. G. K. Batchelor, "Compression waves in a liquid suspension of gas bubbles," Sb. Per. Mekh., No. 3 (1968).
2. V. E. Nakoryakov, V. V. Sobolev, and I. R. Shreiber, "Long-wave perturbations in a gas–liquid mixture," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 5 (1972).
3. L. Von Weingarten, "Unidimensional flows of liquids with gas bubbles," in: Rheology of Suspensions [Russian translation], Mir, Moscow (1975).
4. V. E. Nakoryakov, V. V. Sobolev, and I. R. Shreiber, "Finite-amplitude waves in two-phase systems," in: S. S. Kutateladze (editor), Wave Processes in Two-Phase Media, Inst. Teplofiz. Sib. Otd. Akad. Nauk SSSR, Novosibirsk (1975).
5. R. I. Nigmatulin, A. I. Ivandaev, and A. A. Gubaidullin, "Numerical modelling of transient wave processes in dispersed two-phase media," in: Transactions of the Third International Seminar on Models of Continuum Mechanics [in Russian], Novosibirsk (1976).
6. V. K. Kedrinskii, "Propagation of perturbations in a liquid containing gas bubbles," Zh. Prikl. Mekh. Tekh. Fiz., No. 4 (1968).
7. N. V. Malykh and I. A. Ogorodnikov, "On the use of the Klein–Gordon equation to describe the structure of compression pulses in a liquid with gas bubbles," in: Continuum Dynamics [in Russian], Vol. 29, Inst. Teplofiz. Sib. Otd. Akad. Nauk SSSR, Novosibirsk (1977).
8. V. V. Goncharov, K. A. Naugol'nykh, and S. A. Rybak, "Stationary perturbations in a liquid containing gas bubbles," Zh. Prikl. Mekh. Tekh. Fiz., No. 6 (1976).
9. D. D. Ryutov, "Landau damping analog in the problem of the propagation of a sound wave in a liquid with gas bubbles," Pisma Zh. Eksp. Teor. Fiz., 22, No. 9 (1975).
10. V. I. Karpman, Nonlinear Waves in Dispersive Media [in Russian], Nauka, Moscow (1973).
11. V. G. Gasenko, V. E. Nakoryakov, and I. R. Shreiber, "Burgers–Korteveg–de Vries approximation in the wave dynamics of gas–liquid systems," in: Nonlinear Wave Processes in Two-Phase Media [in Russian], Inst. Teplofiz. Sib. Otd. Akad. Nauk SSSR, Novosibirsk (1977).
12. V. V. Kuznetsov, V. E. Nakoryakov, B. G. Pokusaev, and I. R. Shreiber, "Propagation of perturbations in a gas–liquid mixture," J. Fluid Mech., 85 (1978).
13. V. E. Nakoryakov, B. G. Pokusaev, I. R. Shreiber, V. V. Kuznetsov, and N. V. Malykh, "Experimental study of shock waves in a loquid with gas bubbles," in: Wave Processes in Two-Phase Media, op. cit.

INTERACTION OF SHOCK WAVES IN AN ELASTOPLASTIC MEDIUM WITH HARDENING

V. A. Baskakov

UDC 539.3

A study was made of the laws and the character of the deformation of an elastoplastic material after the passage of shock waves brought about by rather intense sources of perturbations. At a sufficiently great distance from the source, the fronts of the waves in the vicinity of the point of their interaction can be regarded as flat. The model of the medium provides for taking account of two hardening mechanisms [1]: kinematic and isotropic. Using the apparatus of the theory of fractures [2] and the method of [3-5], at first an elastic, and then an elastoplastic self-similar solution of the problem is constructed. The principal difficulty here consists in seeking the previously unknown lines separating the regions of elastic and plastic deformation of the material, at which the boundary conditions are assigned for the solution of a quasilinear system of differential equations in dissipative regions. A study is made of the effect of the hardening parameter on the qualitative side of the interaction of the waves. The basic relations were investigated using a digital computer; concrete numerical results were obtained. The solutions presented are a natural development of [5-7].

Let two flat shock waves in the form of steps Σ_1 and Σ_2 be propagated into an undeformed elastoplastic medium with the velocity G at an angle of $0 < 2\alpha < \pi$ (Fig. 1). Within the framework of the theory of small elastoplastic deformations it is postulated that the total

Voronezh. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 127-134, November-December, 1979. Original article submitted November 24, 1978.

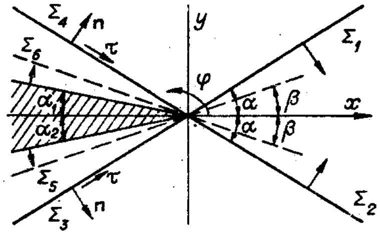

Fig. 1

deformation e_{ij} is made up of elastic e_{ij}^{e} and plastic e_{ij}^{p} parts, and is expressed in terms of the displacements u_i by the Cauchy formulas (i, j = 1, 2, 3). The x_1, x_2, and x_3 axes are orthogonal; all the sought quantities are assumed to be independent of x_3. We seek the solution of the problem in a movable system of coordinates ($x = x_1 - St$, $y = x_2$), connected with the point of interaction of the waves ($S = G(\sin\alpha)^{-1}$, t is the time). In what follows the elastic and neutral regions appearing are called nondissipative, as opposed to plastic regions, in which there is dissipation of energy. In the nondissipative regions, the changes in the stresses and deformations are determined by elastic dependences, while in the plastic regions, the condition of plasticity and the associated law of plastic flow must be brought in.

In the process of the interaction of the waves it may be found that the nondissipative region occupies the whole space behind the starting waves. In the system of coordinates x, y, the field of the stresses, velocities, and deformations will then be stationary behind the fronts of these waves, and the solution can be assumed to be self-similar, i.e., it can be postulated that the components of the tensor of the stresses σ_{ij}, the deformations e_{ij}, and the velocities of the displacement v_i depend only on $\xi \equiv \cot\varphi = xy^{-1}$, where φ is an angle, reckoned from the positive direction of the x axis counterclockwise (thus, $\varphi_+ = +\alpha$ for the wave Σ_1, and $\varphi_- = -\alpha$ for the wave Σ_2 (see Fig. 1).

Using the linear Hooke's law, the Cauchy formulas, and setting $u_1 = yu(\xi)$, $u_2 = yv(\xi)$, $u_3 = yw(\xi)$, we obtain the following system of the equations of motion:

$$(\lambda + 2\mu + \mu\xi^2 - \rho S^2)u'' - (\lambda + \mu)\xi v'' = 0,$$
$$-(\lambda + \mu)\xi u'' + ((\lambda + 2\mu)\xi^2 + \mu - \rho S^2)v'' = 0,$$
$$(\mu(1 + \xi^2) - \rho S^2)w'' = 0,$$

where ρ is the density of the medium; λ, μ are Lamé parameters; the primes denote derivatives with respect to ξ.

The solution of this system is everywhere trivial:

$$u = a\xi + b,\ v = c\xi + d,\ w = e\xi + f,$$

where the determinant is nonzero (a, b, c, d, e, f are constants). A nontrivial solution of the system exists with the condition

$$(\rho G^2 - \mu)^2(\rho G^2 - (\lambda + 2\mu)) = 0,$$

where G is a new variable, determined by the relation $G^2(1 + \xi^2) = S^2$. Thus, in the body there can be propagated both vortexless and shear-type shock waves, respectively, with the velocities $G_1^2 = (\lambda + 2\mu)\rho^{-1}$, $G_2^2 = \mu\rho^{-1}$.

Let us consider the case of the interaction of two vortexless shock waves. In this case, a state of plane deformation is established in the space ($u_3 = 0$); therefore, out of the three equations of motion, there remains only the first two.

1. Construction of Elastic Solution. The determinant of the system of equations of motion is equal to zero with the following values of the angle φ: $\varphi_{1,2}^{*} = \pm\alpha + l\pi$, determining the position of the fronts of longitudinal waves, propagating with the velocity G_1, and $\varphi_{3,4}^{*} \equiv \pm\beta + l\pi = \pm\arcsin\,(\mu/(\lambda + 2\mu))^{1/2}\sin\alpha + l\pi$, determining the position of the fronts of transverse waves, propagating with the velocity G_2. From the condition of the problem posed it follows that, if these waves exist, then, l can be an odd whole number. For definiteness we can set $l = 1$. The position of the vortexless waves Σ_1, Σ_2, Σ_3, Σ_4 are determined by the relationships $\xi_{1,3} = \cot\alpha$, $\xi_{2,4} = -\cot\alpha$, and the shear waves $\Sigma_5\Sigma_6$, by the relationships $\xi_{5,6} = \cot(\pi \pm \beta) = \pm\cot\beta$. However, it can be rigorously proved that, in the present statement, as a result of the interaction of vortexless shock waves, surfaces of a strong discontinuity of Σ_5, Σ_6 are now formed.

In actuality, we postulate the presence of the surfaces Σ_5 and Σ_6, we find the stress–deformation state of the medium in the proximity of the point of interaction of the waves. In what follows, the zone between Σ_1 and Σ_4 will be noted by the number 1, Σ_2, Σ_3 by the number 2, $\Sigma_3\Sigma_5$ by the number 3, $\Sigma_4\Sigma_6$ by the number 4, $\Sigma_5\Sigma_6$ by the number 5. Let the intensities of the starting waves Σ_1 and Σ_2 be equal, respectively, to γ_1 and γ_2. From the conditions of an Adamar set for the normal component of the rate of displacements in zones 1 and 2, we have $V_m = -G_1\gamma_m$ (m denotes the number of the zone). Then, $v_1^{(m)} = V_m \sin\alpha$, $v_2^{(m)} = (-1)^m V_m \cos\alpha$ (here and in what follows summation is not carried out with respect to m). Using the fact that, in a movable system of coordinates, the rates of displacements are expressed by the formulas $v_i = -Su_{i,x}$, there can be deformations, and then, in accordance with Hooke's law, also stresses in these zones:

$$e_{11}^{(m)} = \gamma_m \sin^2\alpha,\ e_{12}^{(m)} = (-1)^m \gamma_m \sin\alpha\cos\alpha,\ e_{22}^{(m)} = \gamma_m \cos^2\alpha,$$
$$\sigma_{11}^{(m)} = \gamma_m(\lambda + 2\mu\sin^2\alpha),\ \sigma_{22}^{(m)} = \gamma_m(\lambda + 2\mu\cos^2\alpha), \tag{1.1}$$
$$\sigma_{12}^{(m)} = (-1)^m \gamma_m \mu \sin 2\alpha,\ \sigma_{33}^{(m)} = \lambda\gamma_m\ (m = 1, 2).$$

Setting $u_1^{(m)} = a_m x + b_m y$, $u_2^{(m)} = c_m x + d_m y$, we obtain the coefficients $a_m = \varkappa_m \sin\alpha$, $c_m = \omega_m \sin\alpha$, where $\varkappa_m = \gamma_m \sin\alpha$, $\omega_m = (-1)^m \gamma_m \cos\alpha$. From the condition of the continuity of the displacements at the surface Σ_1 and Σ_2, we obtain $b_m = (-1)^m \varkappa_m \cos\alpha$, $d_m = (-1)^m \omega_m \cos\alpha$. Thus, in zones 1 and 2, the displacements will be known.

Assuming further that for m = 3, 4, 5 the form of the dependence of the coefficients a_m, b_m, $\varkappa_m$, ω_m is the same as for m = 1, 2, from the condition of continuity of the displacements at the surfaces Σ_3 and Σ_4, we obtain $b_3 = (2\varkappa_2 - \varkappa_3)\cos\alpha$, $d_3 = (2\omega_2 - \omega_3)\cos\alpha$, $b_4 = (\varkappa_4 - 2\varkappa_1)\cos\alpha$, $d_4 = (\omega_4 - 2\omega_1)\cos\alpha$. From the condition of continuity of the displacements at Σ_5 and Σ_6, for the coefficients b_5, d_5 we obtain two expressions, equating which we have the relationships

$$2\varkappa_5 \operatorname{ctg}\beta = (\varkappa_3 + \varkappa_4)(\operatorname{ctg}\beta - \operatorname{ctg}\alpha) + 2(\varkappa_1 + \varkappa_2)\operatorname{ctg}\alpha; \tag{1.2}$$
$$2\omega_5 \operatorname{ctg}\beta = (\omega_3 + \omega_4)(\operatorname{ctg}\beta - \operatorname{ctg}\alpha) + 2(\omega_1 + \omega_2)\operatorname{ctg}\alpha. \tag{1.3}$$

It is well known that, in vortexless shock waves, the tangential component of the displacement rate v_τ is continuous, and, in equivolumetric waves, the normal component v_n. This corresponds to a situation in which, at the first of them $[u_{\tau,n}] = 0$ and, at the second, $[u_{n,n}] = 0$, where the square brackets denote a discontinuity of the given quantities. If now we apply these relationship to the waves Σ_3, Σ_4, Σ_5, Σ_6, previously differentiating the expressions $u_\tau = u_i\tau_i$, $u_n = u_i n_i$ along a normal, after transformations we obtain the following equations:

$$\gamma_2 \sin 2\alpha = \varkappa_3\cos\alpha + \omega_3\sin\alpha,\ \gamma_1\sin 2\alpha = \varkappa_4\cos\alpha - \omega_4\sin\alpha,$$
$$(\varkappa_3 - \varkappa_5)\sin\beta = (\omega_3 - \omega_5)\cos\beta,\ (\varkappa_4 - \varkappa_5)\sin\beta = (\omega_5 - \omega_4)\cos\beta, \tag{1.4}$$

which, together with (1.2), (1.3), form a closed system of linear algebraic equations for $\varkappa_3$, $\varkappa_4$, $\varkappa_5$, ω_3, ω_4, ω_5, and have the solution

$$\varkappa_3 = \varkappa_4 = \varkappa_5 = (\gamma_1 + \gamma_2)\sin\alpha,\ \omega_3 = \omega_4 = \omega_5 = (\gamma_2 - \gamma_1)\cos\alpha. \tag{1.5}$$

From (1.5) specifically, it follows that the solution in zones 3, 4, 5 is identical and is a simple superposition of the solutions in zones 1 and 2. This means that the surfaces Σ_5 and Σ_6 are not present in the packet of waves. In actuality, calculating the components of the vectors of the displacements, and of the tensors of the deformations and the stresses in zones 3, 4, and 5, we have (m = 3, 4, 5)

$$u_1^{(m)} = (\gamma_1 + \gamma_2)x\sin^2\alpha + (\gamma_2 - \gamma_1)y\sin\alpha\cos\alpha,$$
$$u_2^{(m)} = (\gamma_2 - \gamma_1)x\sin\alpha\cos\alpha + (\gamma_1 + \gamma_2)y\cos^2\alpha,$$
$$e_{11}^{(m)} = (\gamma_1 + \gamma_2)\sin^2\alpha;\ e_{22}^{(m)} = (\gamma_1 + \gamma_2)\cos^2\alpha,$$
$$e_{12}^{(m)} = (\gamma_2 - \gamma_1)\sin\alpha\cos\alpha;\ \sigma_{33}^{(m)} = \lambda(\gamma_1 + \gamma_2), \tag{1.6}$$
$$\sigma_{11}^{(m)} = (\gamma_1 + \gamma_2)(\lambda + 2\mu\sin^2\alpha),\ \sigma_{22}^{(m)} = (\gamma_1 + \gamma_2)(\lambda + 2\mu\cos^2\alpha),$$
$$\sigma_{12}^{(m)} = \mu(\gamma_2 - \gamma_1)\sin 2\alpha.$$

From (1.6) it can be seen that the sought quantities do not depend on m.

Thus, the elastic solution obtained, (1.1), (1.6), completes the proof of our assertion. In what follows, the zones 3, 4, 5 will be denoted as a single third zone.

2. Construction of Elastoplastic Solutions. Case of an Ideal Elastoplastic Material. With a determination of this solution, we postulate that, in the body, there exists also a state of plane deformation, and that the material is plastically in compressible. However, in what follows, for brevity in writing we shall continue to use the tensor notation, having in view here only quantities not equal to zero. Let γ_m be such that, in zones 1 and 2, the value of $I_{(m)}$, characterizing the intensity of the stresses, will be equal to $I_{(m)} = 0.5 \cdot S_{ij}^{(m)} S_{ij}^{(m)} = z_m^2 k^2$ (m = 1, 2; S_{ij} are the components of the deviator of the stresses; k is the yield point with pure shear; $0 < z_m \leq 1$). Then, in these zones the solution obtained in Sec. 1 is valid. Under these circumstances, in the third zone, dissipative region can appear only in the case where the waves Σ_3 and Σ_4 become neutral, and the boundaries of these regions are surfaces of a weak discontinuity α_1 and α_2 [4-5] (see Fig. 1). In addition to this, the inequality $I_{(3)} \geq k^2$ must be satisfied; in the contrary case, solution (1.6) holds. Using (1.1) and (1.6) to calculate the intensities in all three zones, we have

$$I_{(m)} = \frac{4}{3} \mu^2 \gamma_m^2 = z_m^2 k^2, \ I_{(3)} = I_{(1)} + I_{(2)} + (2 - 3\sin^2 2\alpha) \sqrt{I_{(1)} I_{(2)}}. \tag{2.1}$$

The above inequality now assumes the form

$$(z_1 + z_2)^2 - 3z_1 z_2 \sin^2 2\alpha \geqslant 1. \tag{2.2}$$

Let (2.2) be satisfied. It is obvious that the plastic fan in the third zone must lie between two neutral regions of this zone. The position of the loading waves $\varphi_1 = \pi - \alpha_1$ is determined from the relation [6]

$$c_1 \sin \alpha - G_1 \sin \alpha_1 = 0, \tag{2.3}$$

where c_1 is the velocity of its propagation, subject to determination.

The continuous solution in regions deformed plastically in the variables x_i, t with the Mises plasticity condition, is described by the equations

$$\begin{gathered} \dot{\sigma}_{ij} = \lambda v_{k,k} \delta_{ij} + \mu (v_{i,j} + v_{j,i} - 2\dot{e}_{ij}^p), \\ \sigma_{ij,j} = \rho \dot{v}_i, \ \sqrt{2} k \dot{e}_{ij}^p = \dot{\varkappa} S_{ij}, \ S_{ij} \dot{S}_{ij} = 0, \end{gathered} \tag{2.4}$$

where $\dot{\varkappa} = (\dot{e}_{ij}^p \dot{e}_{ij}^p)^{1/2} > 0$; the dot denotes the derivative with respect to time; δ_{ij} is the Kronecker symbol. Writing (2.4) at the discontinuities, and using the fact that, at these surfaces Σ_i (i = 1, 2, 3, 4), the plastic deformations are continuous, from geometric and kinematic conditions of a set of the first order [2], for the velocity of the wave α_1, we can obtain

$$2k^2 c_1^2 = A \pm \left(A^2 - 4k^2 G_2^2 \left[(k^2 G_1^2 - G_2^2 B_0^2) - G_3^2 B^2\right]\right)^{1/2}, \tag{2.5}$$

where $A = k^2 G_0^2 - G_2^2 B_0^2$, $G_0^2 = G_1^2 + G_2^2$, $B_0^2 = b_{11}^2 + b_{22}^2$, $G_3^2 = G_1^2 - G_2^2$, $B^2 = (b_{11}\nu_2 - b_{22}\nu_1)^2$, $b_{11} = S_{11}^{(3)}\nu_1 + S_{12}^{(3)}\nu_2$, $b_{22} = S_{12}^{(3)}\nu_1 + S_{22}^{(3)}\nu_2$, $\nu_1 = \sin \alpha_1$, $\nu_2 = \cos \alpha_1$. As follows from (2.5), the velocity of a weak loading wave depends to a considerable degree on the stressed state of the medium ahead of the wave, which is a consequence of the nonlinearity of the starting system of equations. Substituting (2.5) into (2.3), we obtain an equation for determining the position of the loading wave, previously determining the stresses ahead of the wave. For this purpose, we use the relationships

$$[v_i] n_i^{(m)} = \psi_m; \ G_1 [\sigma_{ij}] = -\psi_m \left(\lambda \delta_{ij} + 2\mu n_i^{(m)} n_j^{(m)}\right) \quad (m = 3, 4), \tag{2.6}$$

which must be satisfied at the surfaces Σ_3, Σ_4 (here ψ_m are quantities characterizing the intensities of these waves; n_i are the components of the vector of the unit normal to the corresponding wave). In distinction from the elastic solution, the values of $\psi_m = -G_1\gamma_m$ are determined here from the condition of creep $I_{(3)} = k^2$. Here it can be postulated that σ_{ij}, e^p_{ij}, v_i, depend only on ξ (or on $\varphi = \text{arc cot } \xi$). Then, the system of equations (2.4) goes over into a system of ordinary differential equations. Its trivial solution correspond to a neutral stressed state of the medium. Therefore, the stresses and rates of displacement found from (2.6), as well as the values $e_{ij}^{p(3)} = \varkappa^{(3)} = 0$ are the boundary conditions for obtaining a nontrivial solution of the above system of equations. They are imposed on the surfaces $\varphi_1 = \pi - \alpha_1 (\varphi_2 = \pi + \alpha_2)$. We pass on to their determination. From the second relationship of (2.6), satisfied at the surface Σ_3 and Σ_4, and the condition of plasticity, we obtain, respectively

$$\psi_{3,4} = \frac{3}{4}\frac{G_1}{\mu}\left(D_{2,1} \pm \left(D_{2,1}^2 - \frac{4}{3}\left(z_{2,1}^2 - 1\right)k^2\right)^{1/2}\right). \qquad (2.7)$$

Here

$$D_{1,2} = S_{11}^{(1,2)}\left(\sin^2\alpha - \frac{1}{3}\right) + S_{22}^{(1,2)}\left(\cos^2\alpha - \frac{1}{3}\right) + S_{12}^{(1,2)}\sin 2\alpha - \frac{1}{3}S_{33}^{(1,2)},$$

the values of the stresses $S_{ij}^{(1,2)}$ are calculated from (1.1), where γ_1 and γ_2 are determined now from the first relationship of (2.1). The second root of (2.7) is extraneous, since, e.g., with $z_1 = z_2 = 1$, it reverts to zero, which leads to the absence of the surfaces Σ_3, Σ_4. Thus, the relationships (2.3), (2.5)-(2.7) completely determine the wave α_1 and the boundary conditions for it for the starting system of ordinary differential equations. If the solution is constructed, passing consecutively through the zones 1-3-2, the boundary conditions assume the form

$$\sigma_{ij}^{(3)} = \sigma_{ij}^{(1)} - \psi_4 G_1^{-1}\left(\lambda\delta_{ij} + 2\mu n_i^{(4)} n_j^{(4)}\right),$$
$$v_i^{(3)} = v_i^{(1)} + \psi_4 n_i^{(4)}, \quad \varkappa^{(3)} = e_{ij}^{p(3)} = 0, \qquad (2.8)$$

where $v_i^{(1)}$ are calculated from the elastic solution of the problem, and ψ_4 from (2.7).

If the solution is constructed, passing successively through zones 2-3-1, the boundary conditions assume the form

$$\sigma_{ij}^{(3)} = \sigma_{ij}^{(2)} - \psi_3 G_1^{-1}\left(\lambda\delta_{ij} + 2\mu n_i^{(3)} n_j^{(3)}\right), \quad v_i^{(3)} = v_i^{(2)} + \psi_3 n_i^{(3)}, \quad \varkappa^{(3)} = e_{ij}^{p(3)} = 0,$$

where $v_i^{(2)}$ are determined from the elastic solution of the problem, and ψ_3 from (2.7). In what follows, for definiteness, the first scheme will be used for construction of the solution.

We introduce dimensionless quantities, using the relationships

$$\bar\sigma_{ij} = \sigma_{ij}k^{-1}, \bar e_{ij}^p = \sqrt{2}\mu k^{-1} e_{ij}, \bar\varkappa = \sqrt{2}\mu k^{-1}\varkappa,$$
$$\bar v_i = \left((\lambda + 2\mu)\rho k^{-2}\right)^{1/2} v_i, \qquad (2.9)$$

using which we write the starting system of equations and boundary conditions (2.8). Then, the system of eleven ordinary differential equations with the boundary condition

$$\bar\sigma_{ij}^{(3)} = \bar\sigma_{ij}^{(1)} - \frac{3}{2}\left(\bar D_1 + \sqrt{\bar D_1^2 - \frac{4}{3}\left(z_1^2 - 1\right)}\right)\left(\nu(1-2\nu)^{-1}\delta_{ij} + n_i^{(4)} n_j^{(4)}\right),$$
$$\bar v_i^{(3)} = \bar v_i^{(1)} + \frac{3}{2}\frac{1-\nu}{1-2\nu}\left(\bar D_1 + \sqrt{\bar D_1 - \frac{4}{3}\left(z_1^2 - 1\right)}\right), \quad \bar e_{ij}^{p(3)} = \bar\varkappa^{(3)} = 0 \qquad (2.10)$$

can be solved numerically using one of the known methods, for example, the Runge–Kutta method (ν is the Poisson coefficient). In this case, the above system of equations can be brought into the form necessary for application of this method. We note first of all that the equality $\dot{\bar\varkappa} > 0$, which expresses the condition of the positive character of the rate of dissipation of mechanical energy with plastic deformation of the medium now goes over into the following: $\bar\varkappa' > 0$, in the upper half-plane $(y > 0)$ and $\bar\varkappa' < 0$ at the lower surface $(y < 0)$. Since the system of equations is linear and homogeneous with respect to the derivatives, it is satisfied by the following values: $\bar\sigma'_{ij} = \bar e^{p\prime}_{ij} = \bar v'_{ij} = \bar\varkappa' = 0$, which contradicts the above inequalities. From this it follows that the determinant of the system in the plastic regions should revert to zero. By virtue of this, only ten equations of the system are independent. Since $\bar\varkappa' \neq 0$, all the quantities $\bar\sigma'_{ij}$, $\bar e^{p\prime}_{ij}$, $\bar v'_i$ can be expressed in terms of the value of $\bar\varkappa'$, for which there is a certain freedom of choice. Due to this, the above system of ordinary differential equations will have a nonsingular solution. Therefore, we shall regard the sought solution as limiting for a medium with hardening, where the parameters of the hardening tend toward zero.

3. Construction of an Elastoplastic Solution in a Medium with Hardening. The system of determining equations consists of the first two relationships (2.4) and the equations [6]

$$\sqrt{2}(k + r\varkappa)\dot e_{ij}^p = \left(S_{ij} - qe_{ij}^p\right)\dot\varkappa, \quad \left(S_{ij} - qe_{ij}^p\right)\left(\dot S_{ij} - q\dot e_{ij}^p\right)$$
$$= 2r(k + r\varkappa)\dot\varkappa,$$

where $r > 0$, $q > 0$ are the parameters of the hardening of the material. It is assumed that there are two hardening mechanisms: kinematic and isotropic. The form of the loading surface is determined by multiplication of the first relationship of (3.1) by itself; we have $(S_{ij} - qe^p_{ij})(S_{ij} - qe^p_{ij}) = 2(k + r\varkappa)^2$. The second relationship of (3.1) was obtained by differentiation of the loading surface with respect to the time. Using (2.9), the sought system of equations for the variable φ assumes the form

$$
\begin{gathered}
\bar{\sigma}'_{11} + \bar{v}'_1 \sin\alpha - \nu(1-\nu)^{-1} \operatorname{ctg}\varphi \cdot \sin\alpha \cdot \bar{v}'_2 + \sqrt{2}\,\bar{e}^{p\prime}_{11} = 0, \\
\bar{\sigma}'_{12} + \sin\alpha\,(1-2\nu)(2(1-\nu))^{-1}(\bar{v}'_2 - \operatorname{ctg}\varphi \cdot \bar{v}'_1) + \sqrt{2}\,\bar{e}^{p\prime}_{12} = 0, \\
\bar{\sigma}'_{22} + \nu(1-\nu)^{-1} \sin\alpha \cdot \bar{v}'_1 - \operatorname{ctg}\varphi \cdot \sin\alpha \cdot \bar{v}'_2 + \sqrt{2}\bar{e}^{p\prime}_{22} = 0, \\
\bar{\sigma}'_{33} + \sin\alpha \cdot \nu(1-\nu)^{-1}(\bar{v}'_1 - \operatorname{ctg}\varphi \cdot \bar{v}'_2) + \sqrt{2}\bar{e}^{p\prime}_{33} = 0, \\
(\bar{\sigma}'_{11} - \operatorname{ctg}\varphi \cdot \bar{\sigma}'_{12}) \sin\alpha + \bar{v}'_1 = 0, \quad (\bar{\sigma}'_{12} - \operatorname{ctg}\varphi \cdot \bar{\sigma}'_{22}) \sin\alpha + \bar{v}'_2 = 0, \\
(1 + (\bar{a} - \bar{q})\,\bar{\varkappa})\,\bar{e}^{p\prime}_{11} - (\bar{S}_{11} - \sqrt{2}\bar{q}\bar{e}^p_{11})\,\bar{\varkappa}' = 0, \\
(1 + (\bar{a} - \bar{q})\,\bar{\varkappa})\,\bar{e}^{p\prime}_{12} - (\bar{S}_{12} - \sqrt{2}\bar{q}\bar{e}^p_{12})\,\bar{\varkappa}' = 0, \\
(1 + (\bar{a} - \bar{q})\,\bar{\varkappa})\,\bar{e}^{p\prime}_{22} - (\bar{S}_{22} - \sqrt{2}\bar{q}\bar{e}^p_{22})\,\bar{\varkappa}' = 0, \\
(1 + (\bar{a} - \bar{q})\,\bar{\varkappa})\,\bar{e}^{p\prime}_{33} - (\bar{S}_{33} - \sqrt{2}\bar{q}\bar{e}^n_{33})\,\bar{\varkappa}' = 0, \\
(\bar{S}_{ij} - \sqrt{2}\bar{q}\bar{e}^{p\prime}_{ij})(\bar{S}'_{ij} - \sqrt{2}\bar{q}\bar{e}^{p\prime}_{ij}) - 2(\bar{a} - \bar{q})(1 + (\bar{a} - \bar{q})\,\bar{\varkappa})\,\bar{\varkappa}' = 0,
\end{gathered}
\tag{3.2}
$$

where $\bar{a} = (q + \sqrt{2}r)(2\mu)^{-1} \geq 0$, $\bar{q} = q(2\mu)^{-1}$. We note that, with $\bar{a} = 0$, the system (3.2) determines the stress–deformation state of the medium in the plastic regions of zone 3 in the case of an ideal elastoplastic material.

Thus, we arrive at the Cauchy problem (2.10), (3.2), which must be solved numerically in a digital computer with determined values of a, q, α, ν, z_1, z_2. Here, the unknown boundary α_1, at which the conditions (2.10) are given, is found from (2.3) taking account of (2.5), (2.8), (2.9). However, the relationships (2.5) related to the case of an ideal elastoplastic material; here it is somewhat modified: instead of k^2 we must write $k^2(\bar{a} + 1)$. The sign in (2.5) is selected taking account of the fact that α_1 is referred to the third zone. It can be shown that, for given values of the initial parameters of several values of α_1 this condition is satisfied. Then, each of the roots is verified by numerical integration of system (3.2). Here the initial parameters must satisfy inequality (2.2) from which, specifically, it follows that $\alpha \leq \pi/4$. In each stage of the integration, the condition of a positive character of the rate of dissipation of energy must be verified. The integration is carried until the first condition of (2.6), written in dimensionless form, is satisfied, where m = 3. Then we seek the value of α_2 (weak loading wave) for which this condition is satisfied, after which the rate of propagation of this wave can be found from the relationship $c_2 \sin\alpha = G_1 \sin\alpha_2$. We note that, in the present solution, no account was taken of the possibility of the formation of plastic shock waves, for which $[e^p_{ij}] \neq 0$. We note also [4, 5] that, for such problems, there are still no general theorems of singularity. Therefore, for the solution obtained here, singularity can be shown only with a careful numerical investigation of all the possible solutions, with different values of the parameters: $\bar{a}$, $\bar{q}$, α, ν, z_1, z_2.

For numerical integration of the system of equations (3.2) by the Runge–Kutta method, the sought quantities $\bar{\sigma}'_{ij}$, $\bar{e}'^p_{ij}$, $\bar{v}'_i$ were expressed in terms of $\bar{\varkappa}'$. Since the determinant of the system (3.2) is equal to zero everywhere in the plastic region, it can be differentiated with respect to φ, and a linear expression with respect to the derivatives can be obtained, from which we determine $\bar{\varkappa}' = f(\bar{\sigma}_{ij}, \bar{e}^p_{ij}, \varkappa, \bar{a}, \bar{q}, \nu, \alpha, \varphi)$.

For different combinations of the initial parameters, a table of values of α_1 was obtained, after which the system of equations (3.2) was integrated with a spacing $\Delta\varphi = 0.01$ for each of these values. From an analysis of the results of the numerical calculations it follows, specifically, that the span of the plastic fan $\Delta\alpha = \varphi_2 - \varphi_1$ is constructed with an increase in the hardening parameter $\bar{a}$, independently of ν, α, z_1, z_2. As an illustration, Fig. 2 gives dependences $\Delta\alpha(a)$ for $z_1 = z_2 = 0.8$, $\nu = 0.3$, $\bar{q} = 0$, and for $\alpha = 0.52$, 0.44, 0.35, 0.26 (angles in radians, curves 1-4, respectively). Here, in all the calculations, in relationship (2.5) a — sign was taken; a + sign gives value of $\alpha_1 = \alpha$, which is impossible. In the case of an ideal elasticoplastic material ($\bar{a} = 0$), the calculations were made for the values $z_1 = z_2 = 1$, $\nu = 0.25$, $\alpha = \pi/4$. In this case boundary conditions (2.10) have the

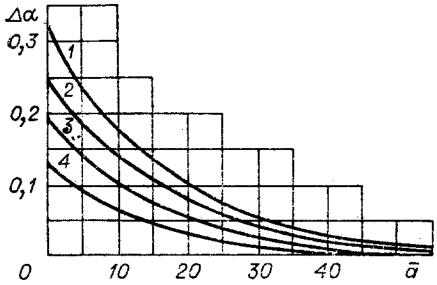

Fig. 2

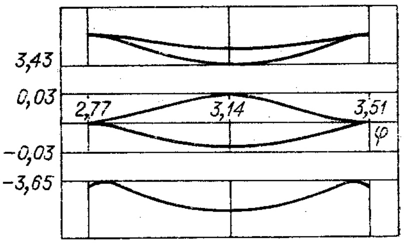

Fig. 3

following values: $\bar{\sigma}_{11} = \bar{\sigma}_{22} = 3.46$, $\bar{\sigma}_{12} = \bar{v}_2 = \bar{e}^p_{ij} = \bar{\varkappa} = 0$, $\bar{\sigma}_{33} = 1.73$, $\bar{v}_1 = -3.675$, $\varphi_1 = 2.77$. The results of numerical calculations for some of sought quantities, characterizing the change in the stress–deformation state of the medium in the plastic fan of the third zone, are shown in Fig. 3 (from top to bottom: $\bar{\sigma}_{22}$, $\bar{\sigma}_{11}$, $\bar{e}^p_{33}$, $\bar{e}^p_{11}$, $\bar{v}_1$). Here, $\varphi_2 = 3.51$, i.e., as was to be expected, the plastic region is disposed symmetrically with relation to the negative x axis. In a more general case ($z_1 \neq z_2$), it can lie as close as desired to this axis.

LITERATURE CITED

1. D. D. Ivlev and G. I. Bykovtsev, The Theory of a Hardened Plastic Body [in Russian], Nauka, Moscow (1971).
2. T. Thomas, Plastic Flow and Fracture in Solids [Russian translation], Mir, Moscow (1964).
3. A. D. Chernyshov, "Reflection of a vortexless shock wave from a rigid wall and the free surface of an elastic half-space. The problem of the motion of a stepwise load with a superseismic velocity at the boundary of the half-space," in: The Dynamics of Continuous Media [in Russian], No. 8, Inst. Gidrodin., Novosibirsk (1971).
4. G. G. Bleich and A. T. Matthews, "Superseismic motion of a stepwise load over the surface of an elastoplastic half-space," in: Mechanics [Russian translation], No. 1 (1968), p. 107.
5. V. A. Baskakov and G. I. Bykovtsev, "Reflection of a plane-polarized wave from a free surface in a hardened elastoplastic medium," Prikl. Mat. Mekh., 35, No. 1 (1971).
6. V. A. Baskakov, "Reflection of vortexless shock waves from the boundary of an elastoplastic half-space," Tr. Inst. Prikl. Mat. Mekh. Voronezh. Gos. Univ., No. 1, Voronezh (1971).
7. V. A. Baskakov, "Effect of hardening on the plastic deformation of a material with the interaction of shock waves with the boundary between two elastoplastic half-spaces," in: Methods of Mathematical Physics in the Mechanics of Structural Continuous Media [in Russian], Vol. 189, VGPI, Voronezh (1976).